Random Surface, Planar Lattice Model, and Conformal Field Theory

Xin Sun

University of Pennsylvania

WHCGP, 1/30/2022

Conformal field theory: QFT with conformal symmtries.

2D CFT: motivated by random surface and 2D lattice model.

- Influenced various branches of mathematics since 1980s: vertex operator algebra, quantum group, moduli space, etc.
- Started to have a major impact on the probablity theory of random surface and lattice model since two decades ago.

Goal for Today

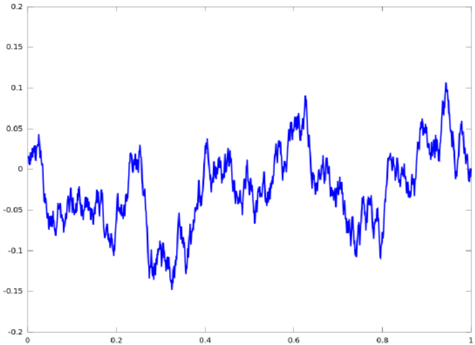
Present a sample of results and open questions in probability that are inspired by CFT ideas and predictions.

Disclaimer: this is not a lecture on CFT itself, but hopefully you want to know more about it afterwards.

How to sample a random path?

Discrete approximation

Scaling limit of simple random walk.

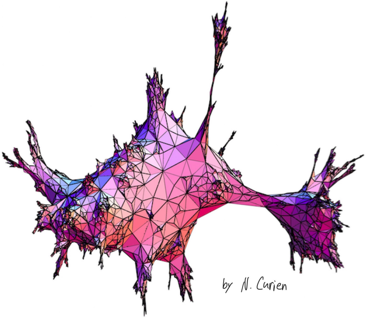


Brownian Bridge

$e_n = \frac{\sqrt{2}}{n\pi}\sin(n\pi t)$. $\{\alpha_n\}$: independent standard Gaussians.

$B = \sum_{n=1}^{\infty} \alpha_n e_n$, (convergence in the uniform topology).

How to sample a random surface?



Discrete approximation

M_n: uniformly sampled triangulation of size n.

Viewed as a piecewise linear Riemannian manifold.

Theorem Le Gall (2011), Miermont (2011)

M_n after proper scaling converge to a random metric measure space in Gromov-Hausdorff-Prokhorov topology.

Brownian sphere: the limiting random sphere.

Random surface and 2D Quantum Gravity

$\mathcal{S}$: a topological surface, e.g. sphere, disk, annulus.

Quantum gravity on $\mathcal{S}$ = random geometry on $\mathcal{S}$.

From random geometry to random function

A random geometry on $\mathcal{S}$, conditioned on being, conformally equivalent to a fixed $(S, \mathbf{g})$, can be written as $(S, e^{\varphi}\mathbf{g})$ for some random conformal factor φ.

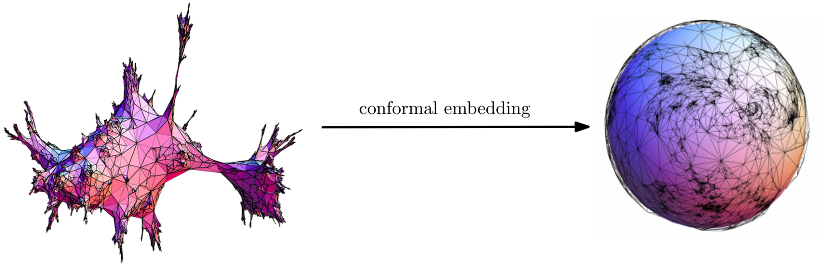
$(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are conformally equivalent if $\exists\ \psi : S_1 \to S_2$ and a function φ on S_2 s.t. $\psi_* \mathbf{g}_1 = e^{\varphi}\mathbf{g}_2$.
ψ: conformal embedding. φ: conformal factor.

CFT description of 2D QG Polyakov (1981)

The conformal factor φ is governed by the Liouville CFT, the 2D quantum field theory defined by Liouville action.

Polyakov's idea in modern probability language

Conformal embedding of Brownian sphere = $\sqrt{8/3}$-LQG on $\mathbb{S}^2$.
The law of the conformal factor φ is given by Liouville CFT.



Liouville quantum gravity (LQG) now have a solid foundation:
random geometry induced by (variants of) Gaussian free field.

Liouville CFT: a particular way of producing variants of GFF.

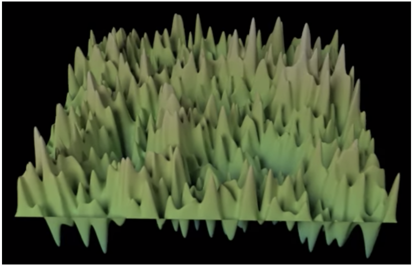
Gaussian Free Field on a Riemannian Manifold

$\{e_n\}_{n\geq 1}$: non-constant eigenfunctions of the Δ on $(S, \mathbf{g})$ normalized by e.g. $\int |\nabla e_n|^2 \mathrm{d}v_{\mathbf{g}} = 2\pi$ and $\int e_n dv_{\mathbf{g}} = 0$.

Gaussian free field (GFF) on $(S, \mathbf{g})$

$$h := \sum_{n=1}^{\infty} \alpha_n e_n, \qquad \{\alpha_n\} \text{ i.i.d. standard Gaussians.}$$

- Convergence holds almost surely in $H^{-1}(S, \mathbf{g})$.
- $\mathbb{E}[h(x)h(y)] = -\log|x-y| + \text{smooth}$.



$h(z)$ is not well defined.

$h_\varepsilon(z)$: average of h over the circle $\{w : |w - z| = \varepsilon\}$.

Simulation of h_ε by H. Jackson.

Random Geometry of γ-LQG

$\gamma \in (0,2)$ φ: a variant of GFF on a planar domain D

γ-LQG area

$A_\varphi^\gamma := e^{\gamma\varphi}\mathrm{d}^2 z := \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma\varphi_\varepsilon}\mathrm{d}^2 z.$

Example of Gaussian multiplicative chaos

Kahane (1985), Duplantier-Sheffield & Rhodes-Vargas, around 2010

γ-LQG boundary length

$L_\varphi^\gamma := e^{\frac{\gamma}{2}\varphi}\mathrm{d}z$ on ∂D. (Gaussian multiplicative chaos)

γ-LQG metric

$d_\varphi^\gamma := e^{\xi_\gamma \varphi}(\mathrm{d}x^2 + \mathrm{d}y^2).$ (more difficult but done)

Dubedat-Ding-Dunlap-Falconet & Gwynne-Miller (2019)

Liouville Conformal Field Theory

Constructed rigorously by making sense of the defining path integral; based on Gaussian multiplicative chaos.

Produce a variant of GFF on each Riemannian manifold (S, g).
sphere: David-Kupiainen-Rhodes-Vargas '14 (original).
disk: Huang-RV '15; annulus: Remy '17 (needed later).
torus: DRV '15; higher genus: Guillarmou-KRV '16.

Integrability of 2D CFT Belavin-Polyakov-Zamolodchikov '84

2D CFT —> local conformal symmetry —> Virasoro algebra —> exact formulae of partition functions/correlation functions.

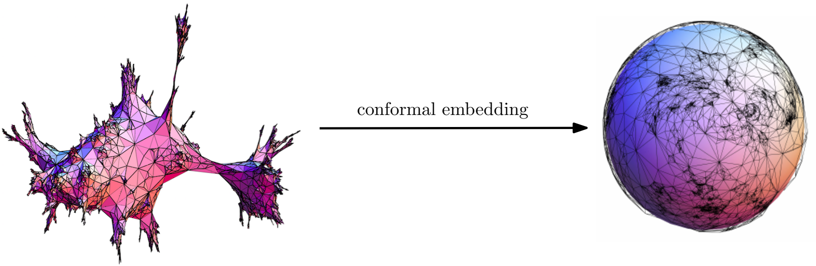
(Rigorous) Integrability of Liouville CFT

DOZZ formula: Kupiainen-Rhodes-Vargas (2017).
Conformal bootstrap: Guillarmou-KRV (2020,2021).

Liouville CFT produce exactly solvable variants of GFF that are relevant to random surface/quantum gravity.

Polyakov's idea in modern probability language

Conformal embedding of Brownian sphere = $\sqrt{8/3}$-LQG on $\mathbb{S}^2$.
The law of the conformal factor φ is given by Liouville CFT.



- Sample φ according to Liouville CFT on $\mathbb{S}^2$. Set $\gamma = \sqrt{8/3}$.
- Then $(\mathbb{S}^2, d_\varphi^\gamma, A_\varphi^\gamma)$ is isometric to the Brownian sphere.
- Uniform triangulations under conformal embedding should converge in the scaling limit to $(\mathbb{S}^2, d_\varphi^\gamma, A_\varphi^\gamma)$.

Theorem Miller-Sheffield '15

One can construct an explicit variant of GFF φ such that $(\mathbb{S}^2, d^\gamma_\varphi, A^\gamma_\varphi)$ is isometric to the Brownian sphere.

Theorem Aru-Huang-S. '17, Ang-Holden-S. '21

Miller-Sheffield variant agrees with Liouville CFT variant on $\mathbb{S}^2$.

A Similar Story for Quantum Gravity on Disk

1. Sample φ according to Liouville CFT on $\mathbb{D}$. Set $\gamma = \sqrt{8/3}$.
2. Then $(\mathbb{D}, d^\gamma_\varphi, A^\gamma_\varphi, L^\gamma_\varphi)$ is isometric to the Brownian disk.
3. Uniform triangulations under conformal embedding should converge in the scaling limit to $(\mathbb{D}, d^\gamma_\varphi, A^\gamma_\varphi, L^\gamma_\varphi)$.

Statements 1 and 2 hold for the disk case, similar to sphere.

The scaling limit conjecture is proved for a discrete variant of conformal embedding. (Holden-S. '19)

Circle packing: a discrete conformal embedding

- Koebe-Andreev-Thurston Circle Packing Theorem
  Triangulations can be uniquely (up to Mobius transforms) represented as tangency relations between circles.
- Rudin-Sullivan (1989):
  Circle packing $\longrightarrow$ Riemann mapping.

In this paper we prove Thurston's conjecture that his scheme converges to the Riemann mapping. Our proof uses a compactness property of circle packings, a length-area inequality for packings, and an approximate rigidity result about large pieces of the regular hexagonal packing (§3 and Appendix 1).

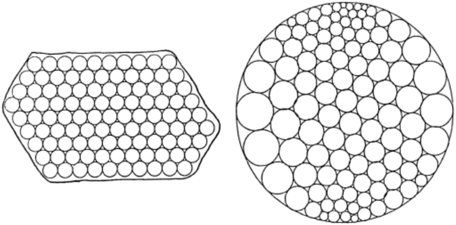
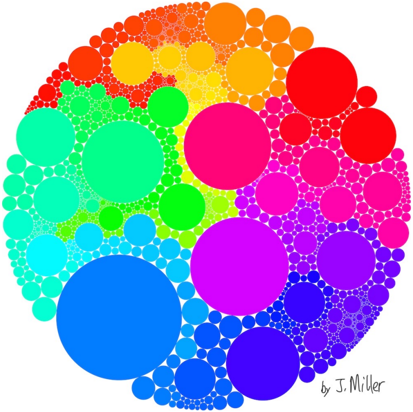


FIGURE 1.1. An approximate conformal mapping

Scaling limit conjecture for discrete uniform random surface

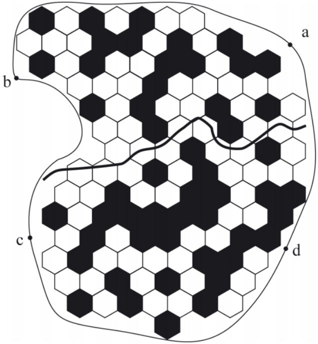
Under various notions of discrete conformal embeddings, uniform triangulation (or quandrangulation, etc.) converge to $\sqrt{8/3}$-LQG, where the field is given by Liouville CFT.



Circle packing case is open.

The only proved case: Cardy-Smirnov embedding
Holden-S. (2019)

(Weaker notions of convergence were proved for various models.)



Ω: Jordan domain.

Site percolation on a piece of triangular lattice restricted to Ω.

Left to right white crossing: a white path separating $\{a, b\}$ and $\{c, d\}$.

δ: side length of the hexagon.
$C_\Omega^\delta(a, b; c, d) := \mathbb{P}[\text{left to right white crossing occurs}]$.

Kesten (1980)

When $p > 1/2$, $\lim_{\delta\to 0} C_\Omega^\delta(a, b; c, d) = 1$.
When $p < 1/2$, $\lim_{\delta\to 0} C_\Omega^\delta(a, b; c, d) = 0$.
When $p = 1/2$, (critical case)
$\liminf_{\delta\to 0} C_\Omega^\delta(a, b; c, d) > 0$ and $\limsup_{\delta\to 0} C_\Omega^\delta(a, b; c, d) < 1$.

Conjecture: Conformal Invariance and Cardy's formula

Aizenmann '91: $\lim_{\delta \to 0} C_\Omega^\delta(a, b; c, d)$ exists and is conformally invariant, which only depends on the cross ratio of (a, b, c, d).

Cardy '92: an exact limiting formula for the case of rectangles based on (non-rigorous) conformal field theory.

Theorem — Smirnov (2001)

Ψ_Ω: the unique conformal mapping from (Ω, a, b, c) to the equilateral triangle $\Delta = \{(x, y, z) : x + y + z = 1\} \cap \mathbb{R}_+^3$.

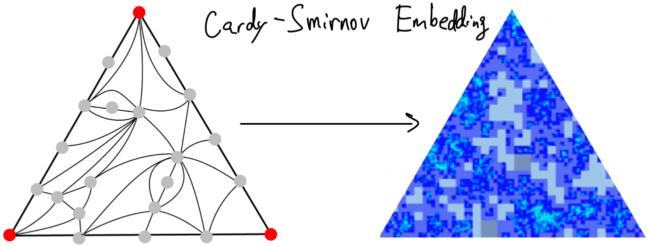
For $z \in \overline{\Omega}$, let $p_a^\delta(z)$ be the probability that there exists a white path separating $\{a, z\}$ and $\{b, c\}$. Similarly define $p_b^\delta(z)$ and $p_c^\delta(z)$.

Then $\lim_{\delta \to 0}(p_a^\delta(z), p_b^\delta(z), p_c^\delta(z)) = \Psi_\Omega(z)$.

$\lim_{\delta \to 0} C_\Omega^\delta(a, b; c, d) = \lim_{\delta \to 0} p_c^\delta(d) = \Psi_\Omega(d)$.
When Ω is rectangle, $\Psi_\Omega(d)$ gives Cardy's formula rigorously.

As Rudin-Sullivan Theorem for circle packing,
Smirnov's Theorem provides a discrete conformal embedding which we call the Cardy-Smirnov embedding.

Holden-S. (2019)



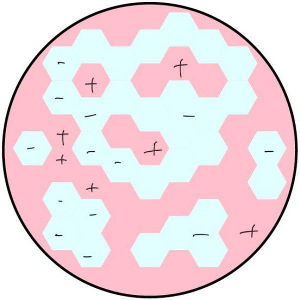
Built on previous joint work with others:

Bernardi-Holden-S. (2018),
Garban-Holden-S.-Sepúlveda (2019),
Albenque-Holden-S. (2019),
Holden-Lawler-Li-S. (2018),
Holden-Li-S. (2018),
Gwynne-Holden-S. (2019).

Many 2D statistical physics models at their criticality enjoys conformal symmetry .

- Partition function $\sim (\det \Delta)^{-\mathfrak{c}/2}$ with $\mathfrak{c} < 1$.
- Correlation functions are governed by a CFT.
- $\mathfrak{c}$: central charge of the corresponding CFT.

Example: 2D Percolation: $\mathfrak{c} = 0$. 2D Ising: $\mathfrak{c} = 1/2$.



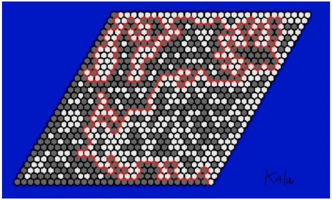
Ising Model on a 2D lattice.

Hamiltonian: $H(\sigma) = \sum_{i \sim j} \sigma_i \sigma_j$.

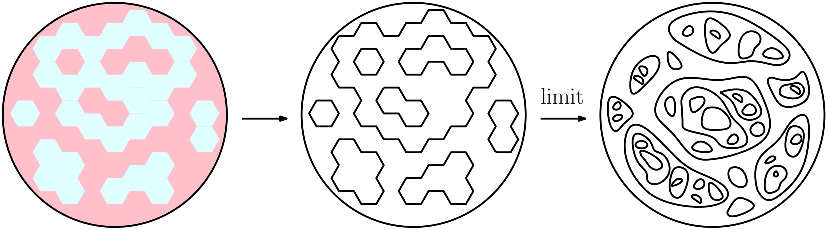
Partition function:
$Z(T) = \sum_\sigma e^{-H(\sigma)/T}$.

$Z(T_c) \sim (\det \Delta)^{-1/4}$.

Schramm Loewner Evolution



Schramm (1999)
Random interfaces in many 2D statical physics models should converge to SLE_κ with $\kappa > 0$.



A few scaling limit results, many more conjectures.

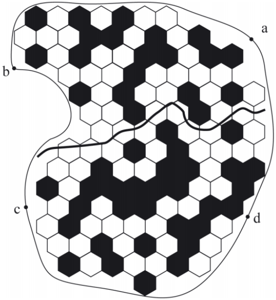
Percolation $\to$ SLE_6, Ising model $\to$ SLE_3 (Smirnov et. al.)

$\mathfrak{c} = 25 - 6(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}})^2$ $\kappa = 6$, $\mathfrak{c} = 0$; $\kappa = 3$, $\mathfrak{c} = \frac{1}{2}$.

Cardy's formula and BPZ equation

Ising model belongs to a well-understood family of CFT called minimal models. [Belavin-Polyakov-Zamolodchikov (1984)]

Percolaion is believed to be described by a CFT. Without understanding it fully, Cardy was able to make predicitions.



Original form of Cardy's formula

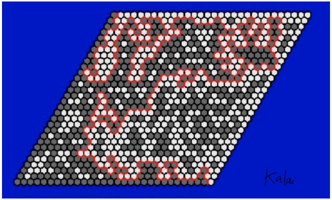
$$\frac{2\Gamma(2/3)}{\Gamma(1/3)^2} \times F(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}, z)$$

z: cross ratio
F: hypergeometric function.

- Viewed as a four-point correlation function of a CFT.
- Solution to 2nd order differential eq. due to BPZ (1984).

CFT and Exact Solvability of SLE

Exact solvability of CFT
$\Longrightarrow$ conjectural formula for scaling limit of lattice models
$\Longrightarrow$ conjectural formula for the corresponding SLE curve.

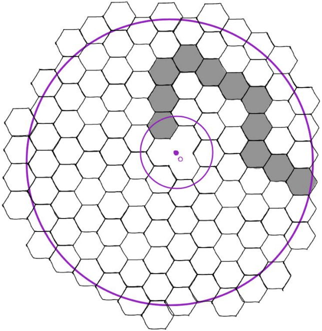


Cardy's percolation formula correpsonds to the formula for the probability of a hitting event for SLE_6 curve.

Many BPZ-equation-inspired formula for SLE can be derived via Itó calculus (martingale observable method) due to the natural connection to 2nd order differential eqs.

Considered by many: Lawler-Schramm-Werner, Kang-Makarov, Dubedat, Zhan, Bauer-Bernard-Kytola, Peltola-Wu,...

Cardy's Formula for Annulus Crossing



Conjecture: Cardy (2006)

$$\lim_{\delta\to 0} \mathbb{P}^\delta[\text{crossing}] = \sqrt{\frac{3}{2}\frac{\eta(6i\tau)\eta(\frac{3}{2}i\tau)}{\eta(2i\tau)\eta(3i\tau)}}.$$

$\tau := (2\pi)^{-1}\log(R/r)$. (modulus)

$\eta(i\tau) := e^{-\frac{\pi\tau}{12}}\prod_{n=1}^{\infty}(1-e^{-2n\pi\tau})$.

(Dedekind eta ftn., ubiquitous in CFT)

- Predicted via a non-rigorous CFT method (Coulomb gas).
- Hard to access by martingale observable method.
- Difficulty: there is no natural notion of time.

Theorem S.-Xu-Zhuang (2022+)

Cardy's conjectural formula for annulus holds.

Proof via the CFT description of quantum gravity on annulus, although the statement is about percolation on lattice.

Brownian Annulus

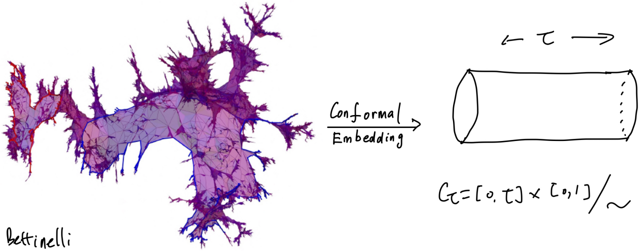
a_n, b_n: even integers with $\lim_{n\to\infty} \frac{a_n}{3n^2} = a$, $\lim_{n\to\infty} \frac{b_n}{3n^2} = b$.
$\mathcal{Q}_n$: set of annular quadrangulations with bdy lengths a_n, b_n.

Sample Q_n from $\mathcal{Q}_n$ with probability $\propto 12^{-\#\text{vertices}}$.

Definition: Brownian annulus with boundary lengths a, b

$\lim_{n\to\infty} Q_n$ in the Gromov-Hausdorff-Prokhorov topology.

Existence follows from work of Betinelli-Miermont.



New question: What's the law of τ?

Theorem (Modulus of the Brownian Annulus) Ang-Remy-S. '22

$$\mathrm{BA}(a,b)^{\#}[\tau \in I] = \int_I {\color{red}\eta(i2\tau)}\rho_\tau\left(\tfrac{b}{a}\right)\,\mathrm{d}\tau, \qquad \forall I \subset (0,\infty).$$

$\mathrm{BA}(a,b)^{\#}$: law of the Brownian annulus with bdy lengths a, b.

ρ_τ: density function for the positive random variable X_τ s.t. $\mathbb{E}[X_\tau^{it}] = \frac{2\pi t e^{-2\pi\tau t^2/3}}{3\sinh(2\pi t/3)}$.

Conjecture Polyakov '81, David '88, Distler-Kawai '89

CFT description of (pure) quantum travity on the annulus:

$$\underbrace{\mathcal{Z}_{\mathrm{GFF}}(\tau)\mathrm{LF}_\tau(\mathrm{d}\varphi)}_{\text{Liouville CFT}} \times \underbrace{\mathcal{Z}_{\mathrm{ghost}}(\tau)}_{\text{ghost CFT}}\; d\tau.$$

$\mathcal{Z}_{\mathrm{GFF}}(\tau) := \frac{1}{\sqrt{2}\eta(2i\tau)}$. $\mathcal{Z}_{\mathrm{ghost}}(\tau) := \eta(2i\tau)^2$.

Special case of the general conjecture for CFT description of Brownian surfaces with non-simply-connected topology.

$\mathrm{BA} = \iint_0^\infty \frac{1}{\sqrt{ab}(a+b)} \mathrm{BA}(a,b)^\#$. (free bdy BA)

Remark: $\frac{1}{\sqrt{ab}(a+b)}$ comes from counting maps in $\mathcal{Q}_m$.
(classical enumeration problem: Tutte, Brown, Bernardi-Fusy, ...)

Theorem Ang-Remy-S. '22

$$\mathrm{BA} = \int_0^\infty (\sqrt{2})^{-1} \eta(2i\tau) \mathrm{LF}_\tau(\mathrm{d}\varphi) \mathrm{d}\tau.$$

LF_τ: pushforward of $\mathbb{P}_\tau \times \mathrm{d}x$ under $(h, x) \mapsto \varphi = h + x$.

$\mathbb{P}_\tau$: law of GFF on $\mathcal{C}_\tau$. $\mathrm{d}x$: Lebesgue measure on $\mathbb{R}$.

$\mathcal{Z}_{\mathrm{GFF}}(\tau) \mathcal{Z}_{\mathrm{ghost}}(\tau) = \eta(2i\tau)/\sqrt{2}$.

Proof outline:

1. $\mathrm{BA} = \int_0^\infty \mathrm{LF}_\tau(\mathrm{d}\varphi)\, m(\mathrm{d}\tau)$ for some measure $m(\mathrm{d}\tau)$.
2. Explicit law of two bdy lengths under $\mathrm{LF}_\tau(\mathrm{d}\varphi)$ for each τ.
   (Integrability of Liouville CFT on annulus by Remy, Wu).
3. Identify $m(d\tau) = (\sqrt{2})^{-1} \eta(2i\tau) \mathrm{d}\tau$ by matching bdy lengths.

Conjecture: CFT description of 2D QG+conformal matter

$$\underbrace{\mathcal{Z}_{\text{matter}}(\tau)}_{\text{matter CFT}} \times \underbrace{\mathcal{Z}_{\text{GFF}}(\tau)\mathrm{LF}_\tau(\mathrm{d}\varphi)}_{\text{Liouville CFT}} \times \underbrace{\mathcal{Z}_{\text{ghost}}(\tau)}_{\text{ghost CFT}}\ d\tau.$$

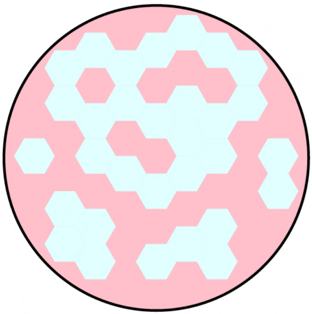
$\mathfrak{c}$: matter central charge. $\mathfrak{c} = 25 - 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$.

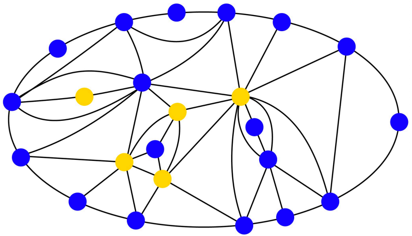
- 2D QG is modeled by random triangulation.
- conformal matter is modeled by lattice models at criticality.
- $\mathfrak{c}$ determines $\gamma \implies$ local LQG geometry.
- $\mathcal{Z}_{\text{matter}}(\tau)$ determines the law of modulus.

Our proof idea for Cardy's annulus crossing formula:

- View annulus-crossed percolation as a $\mathfrak{c} = 0$ matter.
- P[annulus crossing] can be viewed as $\mathcal{Z}_{\text{matter}}(\tau)$.
- Use the same method to get $\mathcal{Z}_{\text{matter}}(\tau)\mathcal{Z}_{\text{GFF}}(\tau)\mathcal{Z}_{\text{ghost}}(\tau)\mathrm{d}\tau$.

Knizhnik-Polyakov-Zamolodchikov (KPZ) Relation





Q: If we have n^2 vertices on the lattice, what is the size of the boundary connecting cluster? Answer: $\sim n^{91/48}$

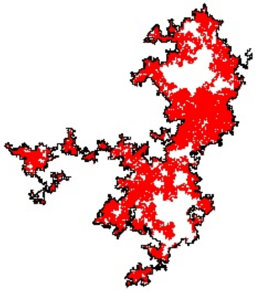
A KPZ derivation of the scaling exponent

1. On random triangulation, the answer is $n^{\text{quantum exponent}}$.
2. $91/48 = \mathrm{KPZ}(\text{quantum exponent})$

Counting maps is "easy"; $\mathrm{KPZ}(\cdot)$ is explicit quadratic.

History of KPZ

- Derived from CFT description of 2D QG+matter. (KPZ '88).
- "Verified" by enumeration of planar maps. (around '90)
  David, Douglas, Gross, Kazakov, Kostov, Migdal, Shenker ...
- Provide a powerful framework to study fractals.



Conjecture Mandelbrot '82

Frontier of planar Brownian motion has fractal dimension $4/3$.

- physics "proof" by KPZ. Duplantier '98.
- rigorous proof via SLE_6.
  Lawler-Schramm-Werner '00.

- 1st rigorous KPZ relation. Duplantier-Sheffield '11.
- KPZ derivation of SLE exp/dim. Gwynne-Holden-Miller '15.

Physics methods for scaling exps/dims of 2D lattice models

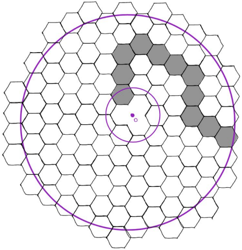
- Exact methods for lattice models. e.g. BPZ '84 for Ising.
- KPZ/quantum gravity method. e.g. Duplantier '98 for BM.

Math methods for proving corresponding SLE results

- Martingale observable method.
- Rigorous KPZ/Liouville quantum gravity method.

Martingale observable method can also give more informative formulae such as Cardy's rectangle crossing formula.

We get such formulae via KPZ/LQG method. (New to physicists?)



$$\mathbb{P}[\text{crossing}] = \sqrt{\frac{3}{2}\frac{\eta(6i\tau)\eta(\frac{3}{2}i\tau)}{\eta(2i\tau)\eta(3i\tau)}}$$

$$\sim (r/R)^{5/48}(1+o(1)).$$

$$\tau := (2\pi)^{-1}\log(R/r).$$

$91/48 = 2 - 5/48.$ (bdy cluster exp)

CFT description of 2D QG+conformal matter on annulus

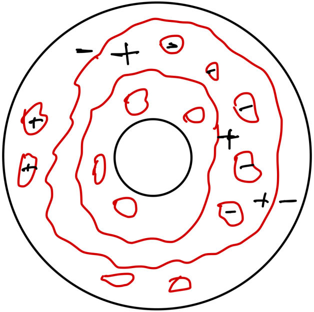
$$\underbrace{\mathcal{Z}_{\mathrm{matter}}(\tau)}_{\text{matter CFT}} \times \underbrace{\mathcal{Z}_{\mathrm{GFF}}(\tau)\mathrm{LF}_\tau(\mathrm{d}\varphi)}_{\text{Liouville CFT}} \times \underbrace{\mathcal{Z}_{\mathrm{ghost}}(\tau)}_{\text{ghost CFT}} \, d\tau.$$

A general quantum gravity method for deriving formulae for lattice models on annulus with SLE as its scaling limits:

1. Interprete the target quantity as $\mathcal{Z}_{\mathrm{matter}}(\tau)$.
2. Define an appropeiate random annulus model.
3. Establish $\int_0^\infty \mathrm{LF}_\tau(\mathrm{d}\varphi)\, m(\mathrm{d}\tau)$ for some measure $m(\mathrm{d}\tau)$.
4. Explicit law of two bdy lengths under $\mathrm{LF}_\tau(\mathrm{d}\varphi)$ for each tau. (Integrability of Liouville CFT on annulus).
5. Solve $m(d\tau)$ hence $\mathcal{Z}_{\mathrm{matter}}(\tau)$ by matching bdy lengths.

General method developed in [Ang-Remy-S. '22].
Application to percolation in [S.-Xu-Zhuang '22+].

An Application to Ising Model



Loop repsentation of Ising model

$Z(T_c) = \sum_{\text{loop collection}} e^{\frac{-1}{T_c}\text{total length}}$

T_c: Ising critical temprature.

$\mathcal{N}$: $\#$ of non-contratible loops

Conjecture: Cardy '06 **Theorem: Ang-Remy-S. '22**

$\lim_{\delta\to 0} \mathbb{E}[n^{\mathcal{N}}] = Z(q,n)/Z(q,1)$. $q = e^{-2\pi\tau} = r/R$.

$$Z(q,n) = \sum_{m\in\mathbb{Z}} \frac{\sin\frac{3(\chi+2m\pi)}{4}}{\sin\chi} q^{\frac{3(\chi+2\pi m)^2}{8\pi^2} - \frac{1}{12}}. \qquad \chi = -\arccos(n/2).$$

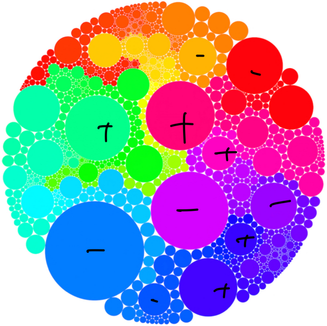
(Again hard to access via martingle observable method.)

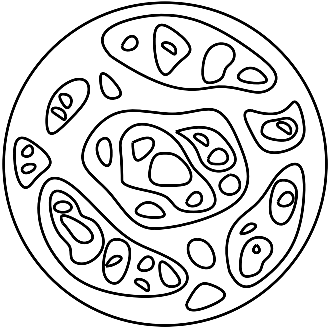
General scaling limit conjecture for the lattice model

$\mathfrak{c}$: central charge of the corresponding CFT.

SLE: $\mathfrak{c} = 25 - 6(\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}})^2$. LQG: $\mathfrak{c} = 25 - 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$.

Ising: $\kappa = 3$, $\mathfrak{c} = \frac{1}{2}$, $\gamma = \sqrt{3}$.





Triangulation + Ising $\Longrightarrow$ $\mathrm{SLE}_3 + \sqrt{3}$-LQG

Unlike triangulation+ percolation (under Cardy-Smirnov embedding),
Scaling limit for the Ising on random triangulation is still open.

Although the scaling limit conjecture is open in most cases, the limiting object: $\mathsf{SLE} + \mathsf{LQG}$ is well understood:

- quantum zipper — Sheffield (2010)
- mating of trees — Duplantier-Miller-Sheffield (2014)

Our quantum gravity method for annulus formula for SLE conceptually relying on counting maps; but in practice we can bypass the scaling limit and directly work in the continuum.

Can give formulae for SLE_κ with $\kappa \neq 3, 6$ as well.

Limitation: so far cannot go beyond annulus.
Crucially rely on the CFT description for Brownian annulus.
Open question for other surfaces, e.g. torus, pair of pants.

Conjecture: CFT description of general Brownian surface

Liouville field on $(S_\tau, g_\tau) \times \mathcal{Z}_{\text{ghost}}(S_\tau, g_\tau)\, d\tau$.
(S_τ, g_τ): a Riemann mainifold with conformal modulus τ.

Ghost CFT: non-physical, come from conformal gauge fixing.

- Central charge of ghost CFT = -26. Polyakov '81
- Liouville central charge: $c_L = 1 + 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$. Polyakov '81
  $\mathfrak{c} + c_L + (-26) = 0 \implies \mathfrak{c} = 25 - 6(\frac{\gamma}{2} + \frac{2}{\gamma})^2$.
- Torus: $\mathcal{Z}_{\text{ghost}}(\tau)\mathrm{d}\tau =$ explicit. Polchinski, David-Rhodes-Vargas
- Surface with genus >1 D'hoker-Phong, Guillamou-R.-V.
  $\mathcal{Z}_{\text{ghost}}(\tau)\mathrm{d}\tau =$ Selberg $\zeta(2)\times$ Weil-Petersson measure.

Difficulties in proving the conjecture beyond annulus:

- No boundary lengths to match for closed surfaces.
- Liouville CFT is exactly solvable but very complicated.
- Lack of fundamental understanding of ghost CFT.

Summary

- CFT gives powerful predictions for random surfaces and 2D lattice models.
- Many of them are verified by SLE/LQG, especially for sphere, disk, and annulus.
- But there are still a lot to be understood.

Outlook

- Convergence of discrete random surfaces to LQG.
- Brownian surfaces beyond sphere, disk, and annulus: random modulus, ghost CFT, Weil-Petersson measure.
- Full understanding of CFT behind percolation: beyond quantum gravity method? rigorous Coulomb gas?
- Many other lattice models: self avoiding walk, dimer, six-vertex, Q-Potts, random cluster, O(n) model...